No. 1
ISSN 0801–9940 May 2008

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RESEARCH REPORT IN MECHANICS

UNIVERSITY OF OSLO
DEPARTMENT OF MATHEMATICS
MECHANICS DIVISION

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Mechanics Divison
DEPT. OF MATH., UNIVERSITY OF OSLO
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ABSTRACT

Present approximate second order methods for the analysis of frames with sway are not capable of reflecting the transition from sway to partly braced, and nearly fully braced behaviour of individual columns in the frames. The main aim of the paper is to extend the approximate storey magnifier approach to account for such a transition. The key to this is in the manner local second order effects are reflected. A high order shear relationship is proposed, and general sway magnifier, critical load and effective length formulations are presented both in terms of first order lateral storey stiffness and critical, free-sway column loads. Their interrelationship, and simplifications leading to existing approaches, and adaptions in present codes and standards, are discussed. Comparisons are made with exact critical loads, sway and moment magnifiers for nearly unbraced, partly braced and nearly fully braced systems.

KEYWORDS

Frames; Columns; Storey-based buckling; Critical load; Effective lengths; Storey magnifier method.

Notation

B_s	Sway magnification factor;
EI, EI_b	Cross-sectional stiffness of columns, and beams;
G_j	Relative rotational restraint flexibility at member end j;
H	Applied lateral storey load (sum of column shears and bracing force);
L, L_b	Lengths of considered column and of restraining beam(s);
N	Axial (normal) force;
N_{cr}	Critical load in general ($= \pi^2 EI/(\beta L)^2$)
N_{cb}, N_{cs}	Critical load of columns considered fully braced, and free-to-sway;
N_E	The Euler buckling load of a pin-ended column ($= \pi^2 EI/L^2$)
R_j	Rotational degree of fixity at member end j;
R_m	Mean rotational degree of fixity of the two member ends;
S_0	First order lateral (storey) stiffness;
S_B	Lateral stiffness of external bracing(s) ;
V_0, V	First order and total (first+second order) shear force in a column;
k_j	Rotational restraint stiffness (spring stiffness) at end j
α_{cr}	Member (system) stability index ($= N/N_{cr}$);
α_b, α_s	Load index of column considered fully braced, and free-to-sway;
α_{ss}	System (storey) stability index
α_E	Nominal load index of a column ($= N/N_E$);
β	Effective length factor (at system instability);
β_b, β_s	Effective length factor corresponding to N_{cb} and N_{cs};
Δ_0, Δ	First order, and total lateral displacement;
γ, γ_n	Flexibility factor in general, and load (N-) dependent flexibility factor;
γ_s, γ_0	Flexibility factor at free-sway, and at zero axial load:
κ_j	Relative rotational restraint stiffness at end j ($=k_j/(EI/L)$).

1 Introduction

It is frequently advantageous to apply approximate methods that are suitable for hand calculation, or easily programmable for spreadsheets etc. They may be used to get solutions of some parts of a problem, or the whole of it, or they may be used as a supplement to more accurate, computer based methods.

In approximate second order analyses of frames with sidesway, the socalled $N - \Delta$ type methods have found extensive use. The basic concept of the initial form of such methods is that the drift and sway moments produced by the vertical (gravity) loads, can be accounted for by equivalent, fictitious lateral loads acting at the beam (floor) levels. The method accounts for the interaction between laterally stiff and flexible columns on the same level, and can be applied to both unbraced or partially braced frames. The method can be applied in an iterative manner by computing total load effects through successive corrections of the first order sidesway displacements. Alternatively, in particular for individual stories, it may be applied in a non-iterative manner based on closed form equations obtained by considering the reduction in lateral column stiffness due to the axial loads. Such storey-based applications are of interest here.

A number of treaties involving the application of various forms of this method to elastic and inelastic structures have been published. Early work (1965-85), dealing with critical loads or second order sway magnification effects, or both, include Rosenblueth [1], Fey [2], Parme [3], Stevens [4], Rubin [5], Horne [6], Wood et al. [7], Hellesland [8], LeMessurier [9], MacGregor and Hage [10] and Lai and MacGregor [11]. Related, more recent work include Aristizabal-Ochoa [12, 13], Lui [14], Xu and Liu[15, 16], Girgin et al. [17], and others, with emphasis on critical load analysis.

A most valuable asset of approximate methods is their transparency with respect to the important variables. This also applies to the manner in which second order effects are reflected. In frames with sidesway, this concerns (1) overall, global ("$N\Delta$") effects, due vertical loads acting on the sidesway of the frame system as such, (2) individual, local ("$N\delta$")effects, due to axial member loads acting on the deflections away from the chord between member ends and thus causing nonlinear (curved) moment distributions along the members, and (3) local effects due to changing restraint stiffness at member ends due to vertical, inter-storey column interaction.

The global second order effects are well taken care of in these approaches. This is to some extent also the case for the local second order ($N\delta$) effects. The latter are generally reflected through a factor with labels such as "bending shape factor" [5], "flexibility factor" [8] or "stiffness reduction factor" [9]. According to the reviewed literature, and textbooks, e.g., [18], it is generally, but incorrectly, stated that the flexibility factor varies between 1 and 1.22 (1.2). This range is normally appropriate for columns in common, regular unbraced frames with columns having similar stiffness and axial load level, with relatively small local second order effects, but it may not be adequate for columns in irregular unbraced frames, or partly braced frames. In such frames, one or more columns may be significantly more flexible than the others due to a combination of length, sectional stiffness and axial load level. Such columns will be effectively braced at a sway imposed by the frame system, and they may significantly affect this sway through their local second order effects. However, these effects on the sidesway are only to a minor extent accounted for by the flexibility factor range indicated above. As a consequence, significant errors in critical load and sway magnification predictions may result.

The emphasis of this study is on such cases for frames with *given* rotational restraints at column ends. Local second order effects on the restraint stiffness distribution due to vertical, inter-storey column interaction, which are of particular interest in single curvature regions of multistorey frames, is considered in more depth elsewhere [19].

The main objective of this paper is to extend the approximate storey magnifier approach for linear elastic two dimensional frames to account for the whole spectrum of sway frame columns, from unbraced to nearly fully braced. This requires the ability of the approach to reflect the interaction between the sway and braced bending (buckling) modes. Towards this goal, (1) the basics of the mechanics of the sidesway displacement and lateral shear interaction of columns are reviewed, (2) general sway magnifier, critical load and effective length formulations are presented both in terms of first order lateral storey stiffness and critical, free-sway column loads, (3) their interrelationship, and simplifications leading to existing approaches, and adaptions in present codes and standards, are discussed, and (4) comparisons are made with exact critical loads, sway and moment magnifiers in nearly unbraced, partly braced and nearly fully braced systems.

2 Load indices, critical and pseudo-critical loads

In describing and discussing the response of columns that may be part of a larger frame, it is often helpful with specific load indices, here defined a priory by

$$\alpha_{cr} = \frac{N}{N_{cr}} \; ; \; \alpha_s = \frac{N}{N_{es}} \; ; \; \alpha_b = \frac{N}{N_{eb}} \; ; \; \alpha_E = \frac{N}{N_E} \qquad (1\,\mathrm{a}-\mathrm{d})$$

where N is the axial column load, N_{cr} is the system (storey) critical load of the column at instability (overall buckling) of the frame system as such, N_{cs} is the critical load of the column if it were free to sway, as calculated with appropriate (most often assumed) rotational restraints, N_{cb} is the corresponding critical load of the column if it were fully braced, and N_E is the "Euler load" (critical load of a of a pin-ended column).

For an elastic framed member of length L, uniform axial load and sectional stiffness EI, these loads can in the conventional manner be defined by

$$N_{cr} = \frac{N_E}{\beta^2} \; ; \; N_{cs} = \frac{N_E}{\beta_s^2} \; ; \; N_{cb} = \frac{N_E}{\beta_b^2} \; ; \; N_E = \frac{\pi^2 EI}{L^2} \qquad (2\,\mathrm{a}-\mathrm{d})$$

where β, β_s and β_b are the respective effective (buckling) length factors. N_{cs} and N_{cb} are simply pseudo-critical loads of the columns, i.e., those that result if the columns are considered in isolation with the "real" (assumed) rotational restraints and with "imagined" lateral restraint conditions (free-sway or fully braced).

The load indices are clearly interrelated. For instance, $\alpha_E = \alpha_s/\beta_s^2$ or $\alpha_E = \alpha_b/\beta_b^2$. The sway buckling load then corresponds to $\alpha_s = 1.0$ or $\alpha_E = 1/\beta_s^2$, and the braced buckling load to $\alpha_b = 1.0$ or $\alpha_E = 1/\beta_b^2$.

3 Transition from sway to braced column

Consider a multibay frame, such as that in Fig. 1, consisting of columns with given rotational end restraints and a lateral bracing modelled by a translational spring with stiffness S_B (force per unit displacement), and subjected to vertical (gravity) loads and to

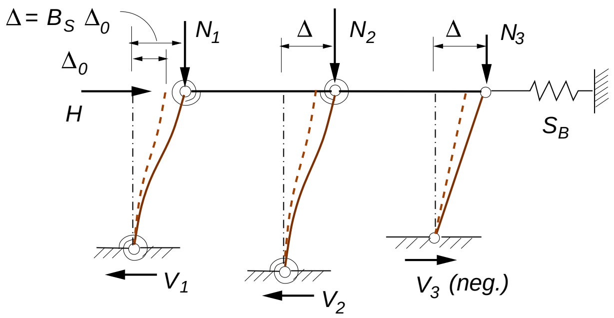

Figure 1: Multibay sway permitted frame.

a lateral load H, applied to the column tops through the connecting beam. It is assumed that the axial deformations in the beams are negligible. The lateral displacement will then be the same in all column axes and in the bracing, and given by

$$\Delta = B_s \Delta_0 \tag{3}$$

where $B_s = \Delta/\Delta_0$ is the storey sway magnification factor, and Δ_0 is the first order displacement due to H. The columns in the frame may act as sway (supporting) or braced (supported) columns, depending on whether they provide or require lateral resistance at some stage of axial loading. With increasing loading the column may change from a sway to a braced column. This transition can best be studied by considering the column shear, which is the key to the transition description.

Approximate shear. For a column in the frame with zero axial load, its shear will become $V = B_s V_0$, where V_0 is the first order shear. Now, if this column was considered in isolation and free to sway, and an axial load was applied, its displacement would increase due to the overturning $N\Delta$ moment. This effect can be accounted for by an equivalent lateral (shear) load $V_{equiv} = \gamma_n N\Delta/L$, where γ_n is a factor that reflects local second order effects (to be discussed in more detail below). For the real column, a lateral reaction equal to this equivalent shear is provided in the opposite direction by the rest of the frame in order to maintain the column's displacement equal to that of the frame. The resulting shear in a framed, axially loaded column, illustrated in the insert on Fig. 2, can therefore be expressed by

$$V = B_s V_0 \left(1 - \frac{\gamma_n N \Delta_0}{V_0 L}\right) \tag{4}$$

The pseudo critical free-sway load can be obtained from Eq. (4) as that causing zero shear. Thus,

$$N_{cs} = \frac{V_0 L}{\gamma_s \Delta_0} \tag{5}$$

where γ_n at the "free-sway" condition (zero shear, α_s=1) is labelled γ_s. This load is a useful parameter in the real framed column description.

Exact shear. The exact shear in an axially loaded, framed column may be given by

$$V = B_v \left(B_s V_0\right) \tag{6}$$

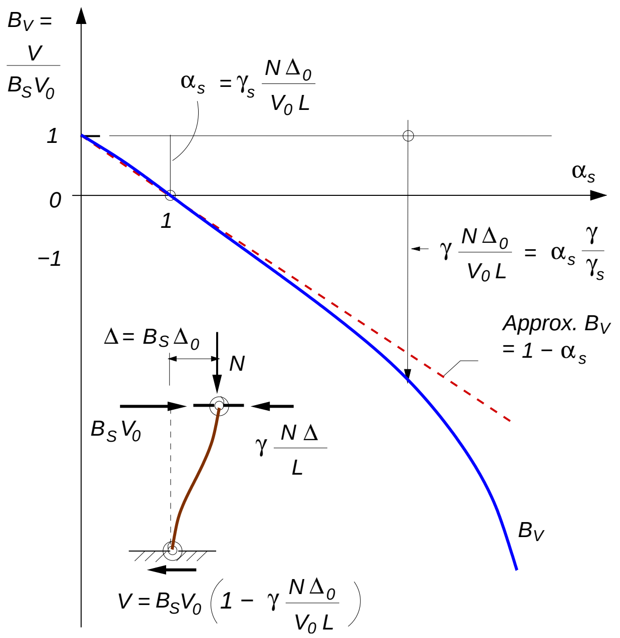

Figure 2: Shear variation with axial load level of a framed column with sidesway restricted to an imposed Δ (by the remainder of the frame).

where B_v is a shear coefficient. With increasing axial loading, the shear must decrease in order to maintain the sidesway at that of the frame, as illustrated in Fig. 2 (solid line) in terms of the free-sway load index. For axial loads in excess of α_s=1 (the free-sway condition), the required shear reverses direction, becomes negative and approaches theoretically minus infinity as the critical braced load is approached.

Supporting and supported column. In a general frame with sidesway, there will be an interaction between flexible columns and stiff columns, and external bracings if present. “Sway” columns with α_s <1 (with positive shear) will contribute to lateral frame stability. Those with α_s >1 (with negative shear) require lateral support by the stiffer columns, and, if present, by external bracings. The latter columns “lean” on the others for lateral stability, and can be considered ”partially braced”. The column hinged at both ends (with V_0=0) is often called a leaning column, and is a special case of the ”partially braced” column category. However, strictly speaking, all laterally supported columns can be considered leaning columns.

Exact γ_n factor. By equating Eq. (4) to (6), and solving for γ_n, exact values can be computed from

$$\gamma_n = (1 - B_v)\,\gamma_s/\alpha_s \tag{7}$$

where the exact shear coefficient B_v can be calculated using the member stiffness relationship with stability functions included, or directly from the differential equation. For instance, for a cantilever column fixed at the base, it can be expressed by

$$B_v = u^3/3(\tan u - u) \tag{8}$$

where $u = \pi\sqrt{\alpha_E}$. Examples of γ_n variations are shown in Fig. 3 for columns with different positive rotational end restraints, by nondimensionless spring stiffnesses (de-

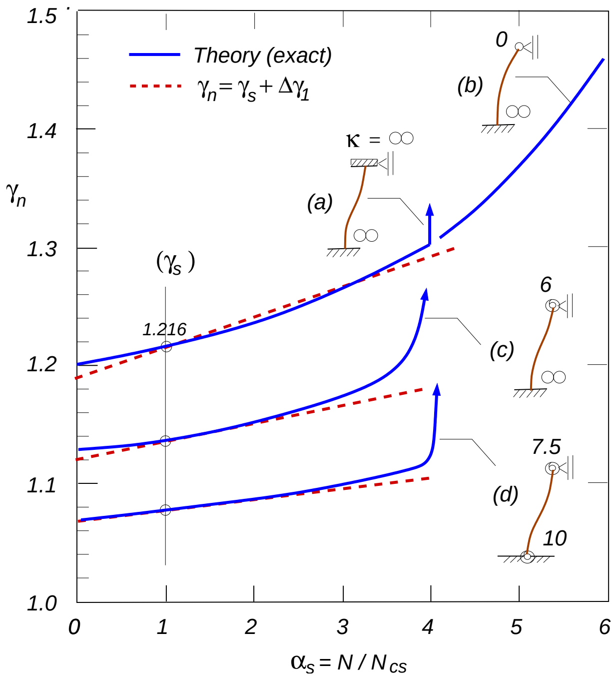

Figure 3: γ_n variation with axial load level.

fined below by Eq. (12)). Curves above the uppermost curve in the figure may result for columns with negative restraints at one of the ends. As the axial load approaches the critical braced load, γ_n approaches infinity rather sharply in cases with equal or nearly equal end restraints (cases (a) and (d)). For cases with larger differences in end restraints, the process is more gradual (case (b)).

4 A general flexibility factor

4.1 Premise

An axial force gives rise to a nonlinear moment distribution along a column, while that of the equivalent shear (discussed above) would be linear. The local second order ($N\delta$) effects, corresponding to the difference between the linear first order and the real nonlinear moment distribution, lead in turn to a reduction in the lateral column stiffness. As briefly stated before, it is this increased flexibility of a column with given rotational end restraints that the γ factor corrects for. It will here, in line with previous practice [8], be labelled "the flexibility factor". It is dependent on the axial (normal) force in the column and will in the general case be denoted γ_n. In many cases, simplified, load independent factors may be justified.

4.2 Proposed γ_n formulation

In order to extend the storey magnifier approach, it is necessary derive a flexibility factor that covers the column response for the whole range of axial loads up to the upper limit represented by the braced critical loads (N_{cb}). A linear approximation beyond α_s=1, that extends the applicability somewhat, has been derived theoretically [15], but the resulting expression is rather cumbersome and and may not be very convenient in practical contexts. Both the linear part, and the further extension into the very nonlinear range, can probably best be described by assuming approximate, reasonably simple relationships.

Based on a study of the mechanics of the response for a variety of parameters, an axial load dependent γ_n approximation is proposed and defined by

$$\gamma_n = \gamma_s + \Delta\gamma_1 + \Delta\gamma_2 \quad (\geq \gamma_s) \tag{9 a}$$

$$\Delta\gamma_1 = 0.12\,(\gamma_s - 1)(\alpha_s - 1) \quad ; \quad \Delta\gamma_2 = q\,\alpha_{s,b}\left(\frac{\alpha_s - 1}{\alpha_{s,b}}\right)^p \tag{9 b, c}$$

Here, $\alpha_s = N/N_{cs}$ is the free-sway load index defined previously, and $\alpha_{s,b}$ is the same index at the fully braced pseudo-critical load,

$$\alpha_{s,b} = \frac{N_{cb}}{N_{cs}} = \left(\frac{\beta_s}{\beta_b}\right)^2 \tag{10}$$

The coefficients p and q vary with rotational end restraints, and in particular with the difference in restraints at the two ends. For the sake of simplicity, fixed values are chosen. Predictions with the combinations

$$q = 1,\ p = 10 \quad \text{and} \quad q = 0.6,\ p = 8$$

have been found to give reasonable agreement with exact results for a wide range of restraints. The latter combination seems to be, on the overall, somewhat better and is chosen here.

For pin-ended columns (N_{cs}=0), γ_n=1. The same value is conservatively recommended for the occasional, rare column with negative axial loads.

Considering that the $\Delta\gamma_2$ contribution, illustrated in Fig. 4 for various exponents p, decreases with increasing exponent, p should strictly increase with decreasing difference in end restraints, to values of about 20-40 for cases with close to the same, or equal, end restraints. This would improve predictions in such cases (discussed further in conjunction with Fig. 8). The increase from $\gamma_n(\alpha_s = 0) = \gamma_0$ at zero axial load, to $\gamma_n(\alpha_s = 1) = \gamma_s$ at the free-sway condition is very minor, Fig. 3. It is certainly acceptable with a constant value at lower load levels. In conjunction with Eq. (9 a), the lower limit given in parenthesis may be adopted when this represents a simplification.

The proposed approximate flexibility factor $\gamma_n = \gamma_s + \Delta\gamma_1 + \Delta\gamma_2$ is valid for any axial load level. Simplifications are justified in a great many practical cases. The simplified, linear

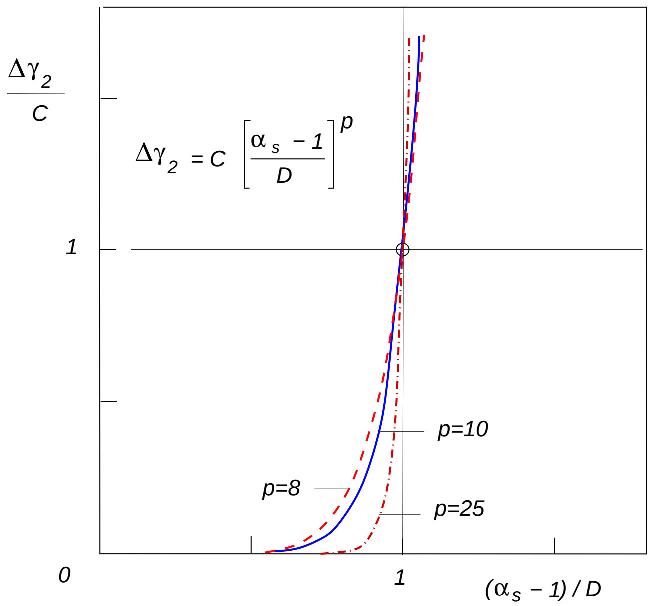

Figure 4: $\Delta\gamma_2$ variation with axial load level.

version $\gamma_n = \gamma_s + \Delta\gamma_1$ (shown by dashed lines in Fig. 3) is valid for low to moderately high load levels for some end restraint combinations, and to very high load levels for columns with nearly equal end restraints. The axial load independent version $\gamma_n = \gamma_s$ may conservatively be adopted for $\alpha_s < 1$. It will also represent an acceptable approximation for low to moderate load levels. The applicability of these will be discussed in more detail later.

4.3 The γ_s factor at free sway

The flexibility factor evaluated at the free-sway condition, $\gamma_n = \gamma_s$, is a parameter in the proposal above. Its variation has been obtained in Hellesland [19] and is shown in Fig. 5 for various combinations of positive and negative end restraint. They are expressed in terms of the smaller (R_{MIN}) and the larger (R_{MAX}) of the **degree of rotational fixity factors** R_1 and R_2 at member ends defined by

$$R_j = \frac{k_j}{k_j + cEI/L} = \frac{1}{1 + c/\kappa_j} \quad \text{with} \quad c = 2 \tag{11}$$

where

$$\kappa_j = \frac{k_j}{EI/L} \qquad j = 1, 2 \tag{12}$$

is the nondimensional rotational restraint stiffness. These fixity factor factors are directly proportional to the first order end moment, or the first order inflection point distance from the end, at which the factor is computed. It is, consequently, a very useful parameter. Its definition evolves naturally from the mathematics of the problem and is closely related to the physics of the column response. At a rotationally fixed end, R=1, and at a pinned end, R=0 (zero fixity). A negative R at an end implies an inflection point

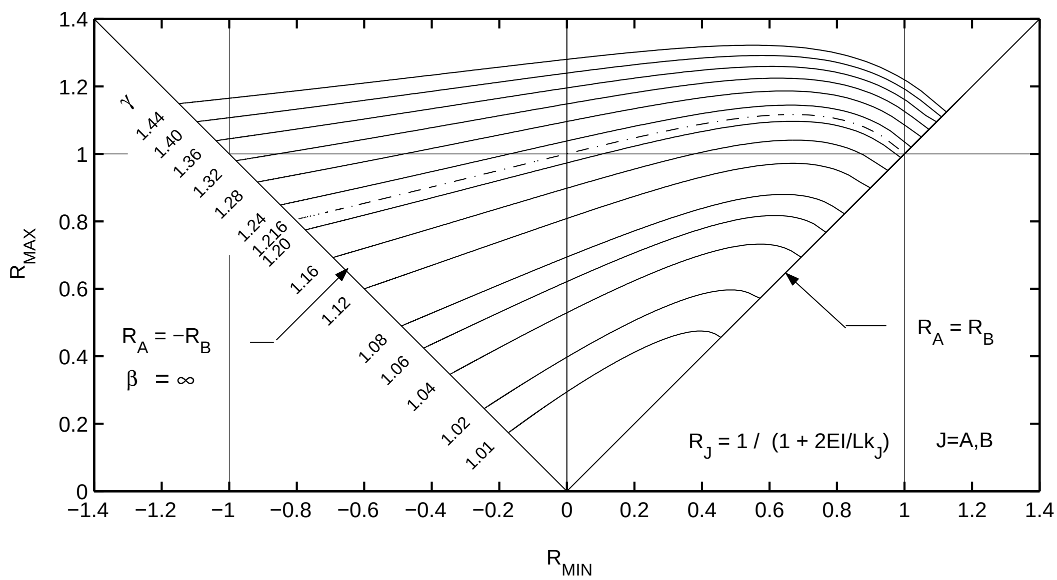

Figure 5: Variation of the flexibility factor $\gamma = \gamma_s$ (at the free-sway condition) in terms of fixity factors of the end restraints (from [19]).

located away from the end, outside the column length. The derivation and additional discussion is available elsewhere [19].

Alternatively, the R factors may be expressed in terms of the well-known G factors that can be defined in a generalised manner by [20]

$$G_j = b_o \frac{(EI/L)}{k_j} = \frac{b_o}{\kappa_j} \qquad j = 1, 2 \tag{13}$$

where b_o is a reference, or scaling, restraint stiffness coefficient taken as b_o=6.

In the figure, $\gamma = \gamma_s$ values have arbitrarily been terminated at 1.44, which is beyond the range of practical interest. Results in the lower right quadrant ($0 < R_{MAX} < 1$ and $0 < R_{MIN} < 1$) and the lower left quadrant ($0 < R_{MAX} < 1$ and $-1 < R_{MIN} < 0$) represent the most common bending shapes. For columns in the lower right, with positive restraints at both ends and double curvature bending, the γ_s values vary between 1 and 1.22 (1.216). In the lower left, they are in single curvature, with negative restraints at one end, and have γ_s values between 1 and about 1.34. The latter case are typical for columns in lower stories of "stiff column-flexible beam" structures. A more extensive discussion is given elsewhere [19].

It is commonly stated in relevant literature, in a general form without any reservations, that the flexibility factor varies between 1 and 1.22 (or 1 and 1.2). This a common misconception. As seen above, this is not correct for γ_s (or γ_0) . It is even less so for the general γ_n value (Fig. 3).

A rather accurate, approximate γ_s factor, that may be written in terms of first order moments and displacements or first order fixity factors (Eq. (11)), or in terms of G

factors (Eq. (13)), can be given by

$$\gamma_s = 1 + \frac{L^4}{167(EI\Delta_0)^2}\left[M_{01}M_{02} + (M_{01} - M_{02})^2\right] \tag{14a}$$

$$\gamma_s = 1 + 0.216\,\frac{R_1R_2 + 4(R_1 - R_2)^2}{(R_1 + R_2 - 3)^2} \tag{14b}$$

$$\gamma_s = 1 + 0.216\,\frac{(G_1 + 3)(G_2 + 3) + 4(G_1 - G_2)^2}{\left[(G_1 + 2)(G_2 + 2) - 1\right]^2} \tag{14c}$$

The basis for these factors, comparisons with exact results, and review of alternative factors are presented in Hellesland [19]. The accuracy of Eq. (14) is very good (normally within a fraction of a percent) for various combinations of positive and negative end restraints.

5 Storey sway magnifiers– Extended approach

5.1 General remarks

Sway magnifiers are derived below. They provide information about the total sidesway, including second order effects. The sway magnifier is normally used also as a moment magnifier for end moments due to lateral loading. In this function, the magnifier is approximate and yields sway-modified first order moments [37]. This will be discussed further (Section 11).

In existing approaches, γ_n is tacitly taken as γ_s or a similar simplification. It is emphasized that both are used, in different contexts, in the extended approach presented below.

5.2 Sway magnifier in terms of first order stiffness

Horizontal frame equilibrium requires that the sum of column shears and the bracing force equals H both in the case without axial loads ($H = S_B\Delta_0 + \sum V_{0,i}$) and in the case with such loads. In the latter case, with the shears V_i in each column "i" defined by Eq. (4) and illustrated in the insert in Fig. 2, horizontal equilibrium gives

$$\begin{aligned} H =& S_B B_s \Delta_0 + \sum B_s V_0 \left(1 - \frac{\gamma_n N \Delta_0}{V_0 L}\right) \\ =& B_s\left(H - \Delta_0 \sum \frac{\gamma_n N}{L}\right) \end{aligned} \tag{15}$$

where the summation is still over all interacting columns, and where subscripts "i" for simplicity are omitted. For columns pinned at both ends (leaning columns), γ_n=1. The same value may conservatively be used in unlikely cases of columns in tension.

The storey sway magnifier, $B_s(= \Delta/\Delta_0)$, can now be solved for from Eq. (15) and given by

$$B_s = \frac{1}{1 - \alpha_{ss}} \tag{16}$$

where

$$\alpha_{ss} = \frac{\sum(\gamma_n N/L)}{S_0} \tag{17 a}$$

and

$$S_0 = \frac{H}{\Delta_0} \quad \text{or} \quad S_0 = \frac{\sum V_0}{\Delta_0} + S_B \tag{17 b, c}$$

Here, α_{ss} is the system (storey) sway stability index, and S_0, in any of the two forms above, is recognised as the first order lateral storey stiffness (force per unit displacement). It includes any external bracings and can in principle also include shear deformations.

B_s above will approach infinity as the loading approaches the critical loading, except when the critical loading is limited by local buckling between ends of a column ($\lim N_{cr}$, Section 8.1). In the latter case, B_s will provide good predictions for loads close to the local buckling inflicted critical loading, but it will approach a finite value rather than the theoretically correct infinity at the local buckling load. This is due to the approximate nature of the assumed γ_n (Eq. (9)), and has no practical consequences.

5.3 Sway magnifier in terms of pseudo-critical loads

The sway magnifier, Eq. (16), may alternatively be computed with a storey stability index defined by critical free-sway loads when shear deformations can be neglected. This is normally the case. The first order flexural lateral stiffness of a column with given rotational end restraints can then be expressed in terms of its free-sway (zero shear) pseudo-critical load, Eq. (5), as $V_0/\Delta_0 = \gamma_s N_{cs}/L$. Then, Eq. (17c) becomes

$$S_0 = \sum \frac{\gamma_s N_{cs}}{L} + S_B \tag{18}$$

and the system stability index, Eq. (17a), can alternatively be written

$$\alpha_{ss} = \frac{\sum(\gamma_n N/L)}{\sum(\gamma_s N_{cs}/L) + S_B} \tag{19}$$

Note that this stability index has the general (extended) load dependent γ_n in the numerator and γ_s (at free sway) in the denominator.

The pseudo-critical loads N_{cs} in the last expression may be computed from the classical expression in Eq. (2b), where β_s may be determined from a wide range of available effective length charts, including the well-known alignment charts and approximate formulas [21] or Eq. (5) for assumed end restraints.

The sway magnifier computed with α_{ss} above will give exactly the same results as the use of Eq. (17) provided they are both based on the same rotational end restraints at the base and top of each column in the summations. Different restraint assumptions, and possible simplifications, will introduce differences. Such aspects are discussed below. Which formulation to use, is often a question of type of application, and sometimes preference.

6 Application considerations

6.1 Rotational end restraints

The rotational end restraints defined explicitly by springs in Fig. 1, are in real frames due to the interaction with beams, and in the case of multistorey frames, also with other columns framing into the considered column end. If a conventional first order analysis has been carried out, and storey stiffness result ($S_0 = H/\Delta_0$) from that analysis is used in the storey stability index, first order rotational restraint stiffness at column ends (k_1 and k_2) will be inherent in the analysis results. They may be given by

$$k_j = f_j k_{bj} \qquad \text{where} \qquad k_{bj} = \sum (bEI_b/L_b)_j \tag{20 a, b}$$

where j=1 and 2, k_{bj} is the restraining stiffness provided by the beams framing into the considered end, b is the bending stiffness coefficient of the beam at the considered end, and, in the case of multistorey frames, f_j is the fraction (or multiple) of the beam restraint that is "allocated" to, or "demanded" by, the column end considered.

The vertical interaction with columns in adjacent storeys is reflected through f_j, given by

$$f_j = \frac{M_{0j}}{(\sum M_0)_j} \tag{21}$$

where M_{0j} is the moment at the considered column end j, and the summation is over this moment and the corresponding moment in the column above, or below. This factor has been given before for braced frames [22], but it is valid for any frame. Moments are defined positive when acting in the same direction (clockwise or anticlockwise). The restraint offered at a joint will in other words be distributed to the columns framing into the joint in proportion to their moments at the joint. In single curvature regions, typical for multistorey frames with stiff columns and flexible beams, f_j will become negative for one of the columns framing into a joint.

The horizontal interaction with neighbouring bays is reflected through k_{bj}. For beams with a semi-rigid connections to the columns (defined by the moment-rotation stiffness relationship $M = k_{co}\psi$), with negligible axial forces and shear deformations, the bending stiffness coefficients of the beams can be given by

$$b = 6\left(\frac{2}{p_n} - \frac{M_f}{M_n}\right)^{-1} \quad \text{or} \quad b = \frac{12\,p_n}{4 - p_n\,p_f}\left(1 + 0.5\,\frac{\theta_f}{\theta_n}\,p_f\right) \tag{22 a, b}$$

where

$$p_i = \frac{k_{co,i}}{k_{co,i} + 3EI_b/L_b} = \left(1 + \frac{3EI_b/L_b}{k_{co,i}}\right)^{-1} \tag{23}$$

Here, $i = n$ (near end) or f (far end), $k_{co,i}$ is the rotational stiffness of the connection assembly itself, M_n and M_f are the beam moments at the near and far end, respectively, θ_n and θ_f are the corresponding joint (column) rotations, and p_i is the degree of rigidity of the connection (hinged: p_i=0; rigid p_i=1).

The beam rotations will be smaller, and equal to $(\theta - \psi)_i$, due to the rotations (ψ) in the connections themselves. By substituting these beam rotations ($i = n, f$) into conventional first order flexibility and stiffness relationships of a beam, as shown for instance in [23], the expressions above can readily be derived. Similar expressions have been given by others (e.g., [15, 18]).

At continuous structural joints, it is normally assumed in analyses that there is no relative rotation between the connected members, provided shear deformations are neglected. I.e., the angle between members connected at the same joint are maintained during moment transfer. The members are then said to be rigidly connected, or in AISC 2005 terminology, that the moment connection is fully restrained (FR). If some relative rotation results, the moment connection is semi-rigid, or partially restrained (PR).

The connection rigidities at continuous joints are p_n=1 and p_f=1. This is the more common analysis case, and yields more familiar bending stiffness expressions [15, 18, 20, 24]. The rotational stiffness of beams, still with neglible axial forces and shear deformations, can for this case be given by

$$b = 4\left(1 + 0.5\,\frac{\theta_f}{\theta_n}\right) \quad \text{or} \quad b = 6\left(2 - \frac{M_f}{M_n}\right)^{-1} \tag{24 a, b}$$

Provided the restraints are computed in the manner described above, the sway magnifier in terms of pseudo-critical loads with Eq. (19), will give identical results to the sway magnifier in terms of first order storey stiffness with Eq. (17).

However, in the sway magnifier approach in terms of pseudo-critical loads (Eq. (19)), end restraints will more often be based on simplified beam restraint assumption such as bending of beams in antisymmetrical, double curvature ($b = 6$), corresponding to an inflection point at the midlength of beams, hinge at the far end ($b = 3$), etc. This will introduce some, but often minor differences between the two approaches in the case of multibay frames, where errors in some bays may be reduced or cancelled by opposite errors in other bays.

Larger differences may result from simplified assumptions about how beam restraints are "shared" between columns of different stories meeting at a joint. For instance by using the f_j factor defined by

$$f_j = \frac{EI/L}{\sum EI/L} \tag{25}$$

which is inherent in the "conventional" G factor definition. This factor, and other similar factors are discussed in [20] in conjunction with system critical load analysis, and there labelled restraint demand factor. A major deficiency is that it does not allow negative values, and thereby implies that beams provide all restraints at a joint. This is far from the case in regions with single curvature bending where the major restraint at a joint is provided by the stiffer column. Despite its deficiencies, this factor is extensively used. Results such obtained may, particularly in critical system load contexts, be improved using the "method of means" [25, 26, 27].

The f_j factor in Eq. (21) should in principle have been defined with total end moments (including local second order effects). These are not known, and Eq. (21) based on first order results represents anyhow the vertical interaction better than Eq. (25) for frames with sway, in particular in regions with single curvature bending. Even so, it may still not be sufficiently accurate in such regions [19].

6.2 First order properties

The storey magnifier approach based on Eq. (17) is most viable in combination with conventional first order frame analyses. However, it may be used even if a first order frame analysis is not carried out. In that case it is necessary to compute the first order lateral stiffness of the columns based on assumed rotational restraints k_1 and k_2 at the ends 1 and 2. For instance, from a convenient expression derived elsewhere [19], by

$$\frac{V_0}{\Delta_0} = c_v \frac{EI}{L^3} \quad ; \quad c_v = \frac{12R_m}{3 - 2R_m} \tag{26 a, b}$$

Here, $R_m = 0.5(R_1 + R_2)$ is the mean of the fixity factors at member ends defined by Eq. (11).

Restated in terms of G-factors (Eq. (13)) with the conventional value for unbraced members of $b_o = 6$, c_v becomes

$$c_v = \frac{12(G_1 + G_2 + 6)}{2G_1G_2 + 4(G_1 + G_2) + 6} \tag{27}$$

Similar expressions can be found in the literature derived along different lines, e.g., [9].

These lateral stiffness expressions may clearly also be used in conjunction with the sway magnifier in terms of pseudo-critical loads, by computing the N_{cs} values using Eq. (5).

7 Simplified storey sway magnifiers

7.1 Simplifications of extended approach

Acceptable storey sway magnifiers can in many cases be obtained with simplified stability indices. For instance, in typical moment frames without excessive difference in axial load levels in the columns, the flexibility factor γ_n in the numerator of Eq. (17) may be approximated by $\gamma_n = \gamma_s$ at the free-sway condition (or γ_0 at the zero axial load level). Then, in the case with S_B=0, Eq. (17) and (19) become

$$\alpha_{ss} = \frac{\sum(\gamma_s N/L)}{(H/\Delta_0)} \quad \text{or} \quad \alpha_{ss} = \frac{\sum(\gamma_s N/L)}{\sum(\gamma_s N_{cs}/L)} \tag{28 a, b}$$

respectively. An advantage of the second formulation is that the mean effect (weighted mean value) of γ_s values in the numerator and denominator in many cases will be about equal and thus cancel out. Unlike the first formulation, the second formulation will then not require the estimation of γ_s factors. In such cases, Eq. (28b) simplifies to

$$\alpha_{ss} = \frac{\sum(N/L)}{\sum(N_{cs}/L)} \quad \text{and} \quad \alpha_{ss} = \frac{\sum N}{\sum N_{cs}} \tag{29a, b}$$

for stories with unequal and equal column lengths, respectively. This simplification is typically acceptable for storeys with reasonably similar and equally loaded columns.

The sway magnifier B_s, Eq. (16), with the two stability index formulations in Eq. (28), and the simplified versions in Eq. (29), were derived and reported in 1976 (Hellesland [8]). At that time, these were rather novel formulations.

At about the same time, in 1977, LeMessurier [9] presented a sway magnifier that can be obtained with Eq. (28a) by introducing constant column lengths, and by replacing the flexibility factor γ_s by $\gamma_s = 1 + C_L$, where C_L was labelled a "correction factor". Thus, with LeMessurier's notation,

$$\alpha_{ss} = \frac{\sum N + \sum(C_L N)}{(HL/\Delta_0)} \tag{30}$$

An alternative expression allowing for unequal column lengths was given in 1992 by Lui [14]. In terms of the stability index, it may be written

$$\alpha_{ss} = \sum\left(\frac{N}{L}\right)\left(\frac{1}{5\sum\eta} + \frac{\Delta_0}{H}\right) \qquad \left(= \frac{\overline{\gamma}_0\sum(N/L)}{(H/\Delta_{0H})}\right) \tag{31}$$

where second order local effects are reflected in the factor η, that is a function of the first order end moment ratio in each column (for details, see [14]). On examining the expression, it is clear that it can be written in almost the same form as Eq. (28a), but with a sort of mean flexibility factor for all the columns in the summation ($\overline{\gamma}_0$), as shown to the right (in the parenthesis) above. In Lui's approach, it is not quite clear how to deal with pin-ended columns.

Another alternative was presented by Aristizabal-Ochoa [12] in 1994 in terms of an effective length factor expression for columns in multibay, unbraced or partially braced frames. On examination of the expression, it is clear that it can be used to establish a stability index that can be given in the exact form as that in Eq. (28a). The flexibility factor has been extracted [19] from Aristizabal-Ochoa's formulation. It is found to give almost the same, but slightly less accurate, values than γ_s given by Eq. (14).

7.2 Design code approaches

The sway magnifier expressed by the pseudo-critical loads (free-sway), as computed with the simplest α_{ss} version above, Eq. (29b), is included in several structural design codes. This includes ACI 318-08 [28], and earlier versions since the 1971 edition, when it was incorporated based on a code proposal by MacGregor et al. [29]. The proposal was based on observation of the storey response. The derivations in the sections above, and earlier [8], show that this intuitive observation is correct when the simplifications discussed above are justified. A similar approach seems to be implied by Yura [30]. The same simplified version is also included in ANSI/AISC 360-05 [24], whereas the version allowing for different column lengths, Eq. (29a), is adopted in AS4100–1990 [31].

Of the two formulations, the one formulated in terms of storey stiffness is probably the most common in codes and standards. The magnifier based on the constant column length version of Eq. (28a) is for instance included in ACI 318-08 [28] and Eurocode 3 [32] with the flexibility factor neglected (γ_s replaced by 1), and in ANSI/AISC 360-05 [24] with a factor giving a constant γ_s=1/0.85=1.18. All three codes limit the application to B_s <1.5. For other cases, more accurate second order analyses are generally required. Eurocode 2 [33] gives a similar procedure for the total structure rather than for individual storeys.

8 Frame (storey) buckling

8.1 Extended approach

In addition to sway and moment magnifiers, critical loads and corresponding effective lengths are normally of interest in second order analyses. The critical loading, at which a structure (system) is on the verge of instability (buckling), is related to a given distribution of initial, reference axial forces (N) in the individual compression members. It can be defined by the critical axial load in each member given by $N_{cr} = \lambda_{cr} N$, where λ_{cr} is the critical load factor, or a set of different, but interrelated, load factors if the vertical loading is not increased by the same factor (nonproportional loading). Here, λ_{cr} is the factor(s) causing infinite sway, or, in other words, that (those) yielding α_{ss}=1.

At system instability in the most common case of proportional loading, which is considered below, the stability index of each column is per definition equal, and equal to the overall storey index, i.e., $\alpha_i = \alpha_{ss}$ $(i = 1, 2, ...)$. The critical load factor in the proportional loading case is the inverse of the stability index ($\lambda_{cr} = 1/\alpha_{ss}$). Thus, an approximate "storey" critical load of column "i" in the storey becomes

$$N_{cr,i} = N_i/\alpha_{ss} \ \leq \lim N_{cr,i} \tag{32}$$

where α_{ss} is given by either Eq. (17) or (19). The corresponding approximate "storey" effective length for a column "i" in the storey (as derived from Eq. (2a)) can be written in any of the two forms given by

$$\beta_i = \left[\frac{N_{E\,i}}{N_i}\,\alpha_{ss}\right]^{1/2} = \left[\frac{N_{E\,i}}{N_i}\,\frac{\sum(\gamma_n N/L)}{S_0}\right]^{1/2} \geq \lim \beta_i \tag{33}$$

where the first order storey stiffness S_0 can be taken according to Eq. (17 b,c) or (18).

Due to the approximate nature of the analysis, it is necessary to impose some restrictions, as indicated by the limits on the expressions above. The need for such limits will be demonstrated later. In a storey (system) instability context, the limits should be based on failure initiated by the "weakest" column failing in a braced buckling mode at a load that may be approximated by N_{cb} (Eq. (2 c)). The weakest column, here taken to be column k, is the one with the larger braced stability index $\alpha_{b,k} = N_k/N_{cb,k}$ (Eq. (1 c)). The requirement is now that the storey stability index should not be taken less than the braced stability index of the weakest column (k), i.e., $\alpha_{ss} \geq \alpha_{b,k}$. This translates to the limits

$$\lim N_{cr,i} = \frac{N_i}{\alpha_{b,k}} \quad \text{and} \quad \lim \beta_i = \beta_{b,k}\left[\frac{\alpha_{E\,k}}{\alpha_{E\,i}}\right]^{1/2} \tag{34 a, b}$$

where $\alpha_{E\,i} = N_i/N_{E\,i}$ and $\alpha_{E\,k} = N_k/N_{E\,k}$ (Eq. (2 d)).

The critical loads, and the effective lengths, of the columns (compression members) of the storey are interrelated through the common α_{ss} factor ($\geq \alpha_{b,k}$). Prediction accuracies will consequently be the same for all the columns, also when the local buckling limits apply. The latter would not be case if the limits in Eq. (34), related to the weakest column, were replaced by $N_{cb,i}$ and $\beta_{b,i}$ of the considered column (i).

8.2 Simplified approach

At lower axial load levels, it is acceptable to replace the axial load dependent γ_n by the load independent γ_s. However, at higher load levels, such predictions may become significantly inaccurate. For this simpler version, it is proposed, as previously suggested [21], to set the upper limit conservatively a few percent greater than β_b to reduce unconservative errors for individual columns near the fully braced condition. Then, in simplified applications, Eq. (32) and (33) should be replaced by

$$N_{cr,i} = N_i/\alpha_{ss} \quad \leq \lim N_{cr,i}/a^2 \tag{35}$$

$$\beta_i = \left[\frac{N_{E\,i}}{N_i}\,\frac{\sum(\gamma_s N/L)}{S_0}\right]^{1/2} \geq a \cdot \lim \beta_i \tag{36}$$

where a=1.05 is suggested here for columns with reasonably equal end restraints, and a=1.1 for columns with very different, yet still realistic, end restraints.

With such a-values, maximum unconservative errors in critical load predictions will be less than about -6% to -11% for practical restraint cases (with no ideal hinge or full fixity). At axial load levels below $0.6N_{cb,}$, the accuracy will be improved, and less than about -2% to -4%, which is generally very acceptable. Additional details are given below (Section 9).

Hajjar and White [34] have studied the accuracy of a number of similar simplified approaches, and also compared to lower limits on effective lengths (upper limits on critical loads) imposed by the AISC code of 1993. These limits, expressed in terms of free-sway properties and also retained in the present code [24], are considerably more conservative than the limits suggested above. The reason for the very conservative AISC limit is not clear to the author.

8.3 Free-sway effective length factors

For the unbraced, free-sway case, $\gamma_n = \gamma_s$ is the correct value. For this case, the effective length of an elastic member with constant cross-sectional bending stiffness and axial load along the length, is obtained from Eq. (36) as

$$\beta_s = \left[\frac{\gamma_s N_E}{S_0 L}\right]^{1/2} \quad \text{or} \quad \beta_s = \left[\frac{\gamma_s \pi^2}{12}\,(\frac{3}{R_m} - 2)\right]^{1/2} \tag{37a, b}$$

in terms of the mean rotational degree of fixity factor, $R_m = 0.5(R_1 + R_2)$.

These free-sway effective length factor expressions, and also one expressed in terms of G factors, have been compared to exact results for a wide range of positive and negative restraint combinations [19]. For combinations of positive restraints, results are within 0.1% of exact results. Also for other restraint combinations, the results are very good and normally exact to two decimals.

9 Critical load predictions

The column defined in Fig. 6 with rotational and lateral (translational) spring restraints is considered. The column might be an isolated column and the lateral spring restraint may represent an external bracing assembly of some kind with stiffness S_B. Alternatively, the column may be a part of an unbraced multibay frame, in which case the lateral spring

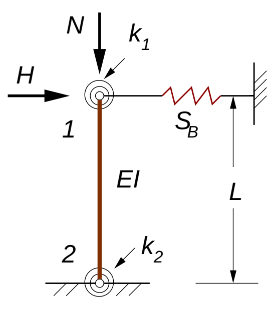

Figure 6: Rotational restrained, partly braced column used in calculations.

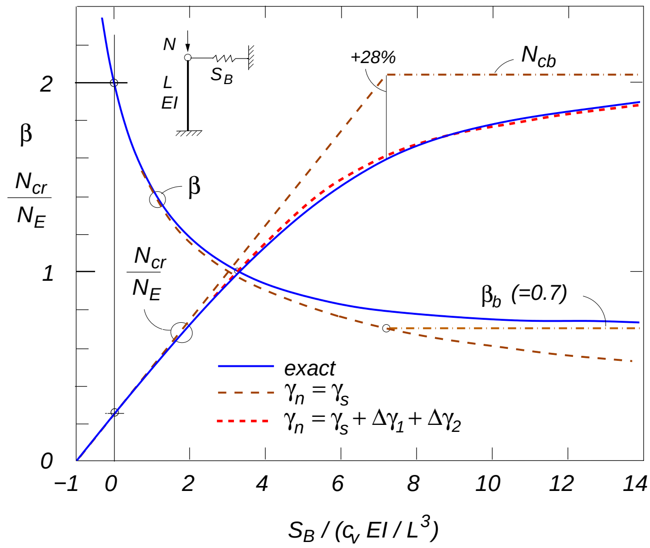

Figure 7: Critical load and effective length according to exact and approximate theories ($p = 8, q = 0.6$; $c_v = 3$) versus lateral bracing stiffness.

stiffness represents the lateral stiffness of the columns of the rest of the frame ($S_{B,col}$), i.e. without the the considered column "j" included. Or it may be a combination of both ($S_B = S_{B,ext} + S_{B,col}$). The contribution of the other columns, but the one considered, to the lateral stiffness can be expressed [35] by

$$S_{B,col} = \frac{H}{\Delta_0} \cdot \frac{(\gamma_n N/L)_j}{\sum(\gamma_n N/L)} - \frac{V_{0j}}{\Delta_0} \tag{38}$$

in which the summation is over all columns, included the one considered.

For the system in Fig. 6, the total first order stiffness is $S_0 = (V_0/\Delta_0) + S_B$, where the first term may be expressed by Eq. (26). Then, with α_{ss} given by Eq. (17a) as

$$\alpha_{ss} = \frac{(\gamma_n N/L)}{S_0} \quad \text{with} \quad S_0 = c_v \frac{EI}{L^3} + S_B \tag{39a, b}$$

the critical load can be given by

$$N_{cr} = \frac{N}{\alpha_{ss}} = \frac{1}{\gamma_n} \cdot \frac{c_v EI}{L^2} (1 + \overline{S}_B) \quad \text{with} \quad \overline{S}_B = \frac{S_B}{(c_v EI/L^3)} \tag{40a, b}$$

Exact critical load results, computed from a convenient effective length expression derived by Cheong-Siat-Moy [36], and predictions by Eq. (40) for the simplified γ approximation $\gamma_n = \gamma_s$ and the extended, load dependent approximation, Eq. (9), are shown in Figs. 7 and 8. Critical loads are zero for $\overline{S}_B = -1$, i.e., when the lateral bracing stiffness is equal and opposite to the columns own first order lateral stiffness.

Also included in the figures are the exact, fully braced critical loads, N_{cb} (horizontal lines), that form upper limits on the other results. For columns with different rotational end restraints, the critical load approaches the fully braced critical load asymptotically with increasing lateral restraint. Columns with equal end restraints will be fully braced

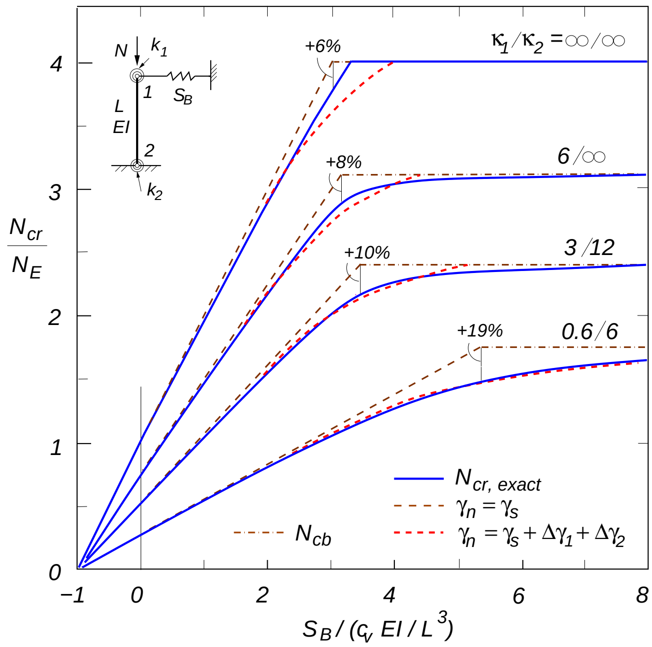

Figure 8: Critical load according to exact and approximate theories ($p = 8, q = 0.6$) versus lateral bracing stiffness.

at a finite bracing value S_B, as seen for the upper case in Fig. 8. Also columns with reasonably equal end restraints may, for practical purposes, be considered braced at finite values of S_B at which N_{cr} is nearly equal to N_{cb}.

Extended approach: $\gamma_n = \gamma_n(N_{cr})$. The extended γ_n, Eq. (9), is a nonlinear function of the axial load, and more specifically, the axial load at the stage of interest, which in the present context is the critical load stage. Thus, $\gamma_n = \gamma_n(N_{cr})$, and an explicit solution for N_{cr} from Eq. (40) is not feasible. For a given S_B, it is therefore necessary to obtain N_{cr} by iteration. Alternatively, and simpler, S_B can be solved explicitly from Eq. (40) for assumed critical loads until a computed S_B is sufficiently close to the given S_B.

The results in the figures are obtained by specifying N_{cr} and computing the corresponding S_B. The agreement with the exact critical results is generally seen to be very good, and generally well within $\pm$ 2% ($\pm$ 1% in corresponding effective lengths), for the various rotational end restraint combinations.

Predictions for columns with equal (upper case in Fig. 8), or nearly equal end restraints, will become somewhat conservative close to the braced critical load (at most about$-$6%). This conservativeness could have been reduced, as mentioned before, by given larger exponents p (in Eq. (9)) in such cases, thereby reducing the $\Delta\gamma_2$ contribution to γ_n.

Simplified approach: γ_n approximated by γ_s. For this simplified case, it can be seen from Eq. (40) that the critical load increases linearly with increasing bracing stiffness S_B. Such predictions are shown in Figs. 7 and 8 by the straight, inclined lines.

The error in these predictions increase with increasing difference in rotational end restraints, and with increasing axial load level up to a maximum at the intersection with the horizontal, "fully braced" lines (N_{cb}). Maximum unconservative errors are indicated in the figures and range from about +28% in the pinned/clamped case (Fig. 7) to +6% for the clamped/clamped case ((Fig. 8)). In corresponding effective length predictions, shown in Fig. 7 only, the respective errors would be between about −13% and −2%.

The pinned/clamped column with the greatest error, represents a severe test case, but is probably mostly of academic interest since ideal fixity and an ideal hinge are not easy to attain in practice. The column in Fig. 8 with κ_1=0.6 (G=10) and κ_2=6 (G=1) represents probably a more realistic "extreme" restraint-difference case. With this considered the extreme case, maximum errors in critical loads are reduced from about +28 to about +19% (-8% in effective length). If the upper limit N_{cb}/a^2 in Eq. (35) is imposed, the maximum unconservative error is further reduced from about +19% to about +14 % (−7% in effective length) with a=1.05, and to about +11 % (−5% in effective length) with a=1.1.

For low to moderate axial load levels, the accuracy of the simplified approach is very good. Even for a relatively high axial load of $0.6N_{cb}$, which is not exceeded in a great many practical applications (and which would correspond to an upper limit obtained with a=1.29), the unconservative error in the critical load does not exceed about +4% (−2% in effective length) even in the extreme case discussed above. For more practical restraint differences, the error is smaller.

By neglecting local second order effects altogether, by replacing γ_s by 1.0, results may become quite inaccurate at all load levels, except for flexibly restrained columns for which γ_s is close to 1.0.

10 Sway magnifier predictions

Sway magnifiers are studied for the system consisting of a laterally partly braced cantilever column, fully fixed at the base and pinned at the top (Fig. 6, $k_1 = 0, k_2 = \infty$). Results for this case are more sensitive to approximations in local, second order member effects ($N\delta$) than similar cases with smaller differences in rotational end restraints. It is therefore a good case for testing the applicability of the approximate methods.

For this system, the exact elastic sway magnification factors can be expressed explicitly, as the ratio between the first order and the total (exact) lateral stiffness, by

$$B_{s,exact} = \frac{S_0}{S_{exact}} = \frac{1 + \overline{S}_B}{B_v + \overline{S}_B} \tag{41}$$

where B_v is given previously by Eq. (8).

Four different approximate sway magnifiers $B_s = 1/(1 - \alpha)$ are considered. These are: (1) the "simple storey magnifier" $B_{s,o}$; $\alpha = \alpha_{ss}$ (Eq. (39)), with $\gamma_n = 1$.

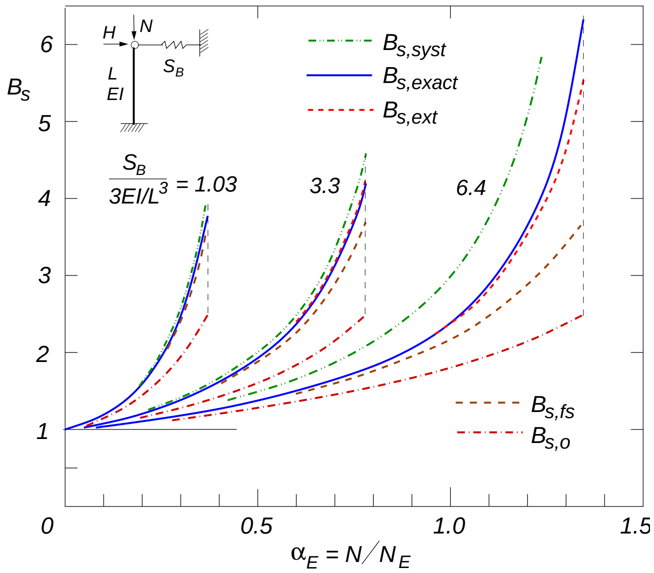

Figure 9: Sway magnifier according to exact and approximate theories ($p = 8, q = 0.6$; $c_v = 3$) versus axial load level, for various lateral bracing stiffnesses.

(2) the "common storey magnifier" $B_{s,fs}$; $\alpha = \alpha_{ss}$ (Eq. (39)), with $\gamma_n = \gamma_s$ (Eq. (14)).
(3) the "extended storey magnifier" $B_{s,ext}$; $\alpha = \alpha_{ss}$ (Eq. (39)), with $\gamma_n = \gamma_s + \Delta\gamma_1 + \Delta\gamma_2$ (Eq. (9), $q = 0.6, p = 8$).
(4) the "system magnifier" $B_{s,syst}$; $\alpha = \alpha_{cr} = N/N_{cr,exact}$ defined with the exact system critical load.

These sway magnifiers are compared to exact results in Fig. 9 and 10 versus the nominal load index $\alpha_E = N/N_E$ (Eq. (1)). Four lateral bracing stiffnesses are considered: $\overline{S}_B$=1.03, 3.3 and 6.4 in Fig. 9, and 34 in Fig. 10. They range from very flexible to very stiff. The corresponding system critical load indices are (to one decimal) $\alpha_{E,cr} = N_{cr,exact}/N_E$=0.5, 1.0, 1.5 and 2.0, respectively. These correspond to 2, 4, 6 and 8 times the free-sway load index of $\alpha_{E,s} = N_{cs}/N_E$=0.25 ($\beta_s = 2$), and to about 0.25, 0.49, 0.74 and 0.98 times the braced load index of $\alpha_{E,b} = N_{cb}/N_E$=2.04. The curves are arbitrarily terminated at the load level at which the "simple storey magnifier" $B_{s,o}$=2.5.

Due to the lateral bracing, there will be an interaction between the column's sway and fully braced bending mode. With increasing bracing stiffness, the column response will to an increasing extent be dominated by the fully braced response. The predictions based on the proposed "extended storey magnifier" are seen to be in very good agreement with exact results, and consequently reflect this increasing interaction well.

The "common storey magnifier" predictions, $B_{s,fs}$ with γ_n replaced by γ_s at free sway ("fs"), are adequate up to rather high axial load levels in cases with flexible lateral bracing ($\overline{S}_B$=1.03 and 3.3). They become increasingly inadequate as the influence of the braced buckling mode become increasingly significant with increasing bracing stiffness ($\overline{S}_B$=6.4 and 34 in the figures). However, even in such cases they are acceptable at low and intermediate axial load levels less than about 60-70% of the critical (system) loads.

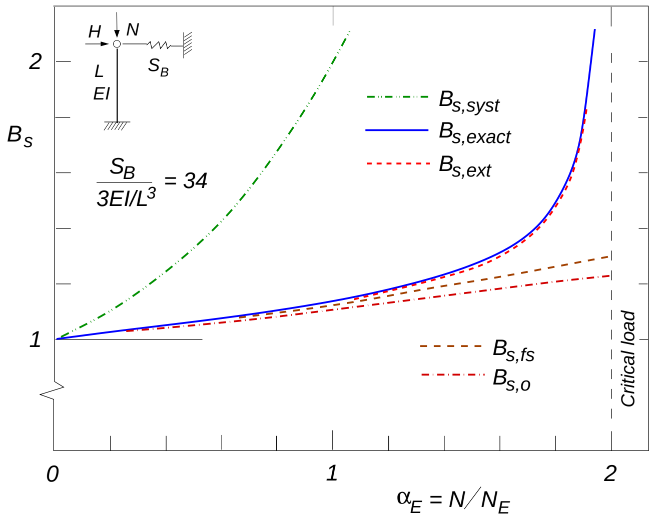

Figure 10: Sway magnifier according to exact and approximate theories ($p = 8, q = 0.6$; $c_v = 3$) versus axial load level, for a very stiff lateral bracing.

This is far from the case with the "simple magnifier" predictions, $B_{s,0}$. Use of γ_n replaced by 1 is generally quite unconservative, and unacceptable, in all cases considered of this particular column. For other cases with more flexible rotational end restraints, the discrepancy will be reduced and γ_n replaced by the numeral 1 will be more acceptable.

The "system magnifier" $B_{s,syst}$, based on α_{cr} defined with the exact critical load, implies that the sway-braced buckling interaction at the critical load is present at all load levels. The predictions are acceptable for the low bracing stiffnesses ($\overline{S}_B$=1.03 and 3.3), but become increasingly, and excessively, conservative as the bracing stiffness, and the interaction with the braced buckling mode, increases and approach the fully braced case.

For bracing giving nearly fully braced critical loads, it is normally expected that sway is negligible. This is not so in the considered cantilever column case in Fig. 10 with the very stiff lateral restraint of $\overline{S}_B = 34$. The critical load index of this column ($\alpha_{E,cr} = 2.0$) is about 98% of the fully braced critical load index of $\alpha_E = 2.04$. Even so, in this nearly fully braced case, there is a significant sway magnification of about 20% at 60% of the critical load ($\alpha_E = 1.2$), which is not an unrealistic load level in practice.

As mentioned previously (Section 7.2) many codes allow the use of $B_{s,o}$ and $B_{s,fs}$ provided these factors do not exceed 1.5. For higher values, the accuracy is not considered satisfactory, and more accurate methods are required. For the cases considered in Fig. 9 and 10, it is clear that this limit (on $B_{s,o}$ and $B_{s,fs}$) is not a suitable limit for indicating acceptable accuracy. It is believed that a limit in terms of the critical load would be more appropriate. For the present results, 60-70% of the critical (system) loading would seem to be a reasonably limit in conjunction with $B_{s,fs}$.

11 Moment predictions

The sway magnifier is normally used also as a moment magnifier for first order end moments due to lateral loading (M_0). The resulting moments, $B_s M_0$, may be termed sway-modified first order moments [37], and are correct for axially not-loaded columns with linear moment variations along the lengths, but not for for axially loaded columns. In the latter, the local (member) second order effects ($N\delta$) will cause a nonlinear moment distribution along the column length and may develop a maximum moment between column ends. The end moments and maximum moment due to a lateral load on an axially loaded column may be expressed by

$$M_j = B_j(B_s M_{0j}) \qquad (j = 1, 2) \tag{42}$$

$$M_{max} = B_{max}(B_s M_{02}) \tag{43}$$

where B_j is the end moment multiplier, B_{max} the maximum moment multiplier, and M_{02} the larger of the two first order end moments.

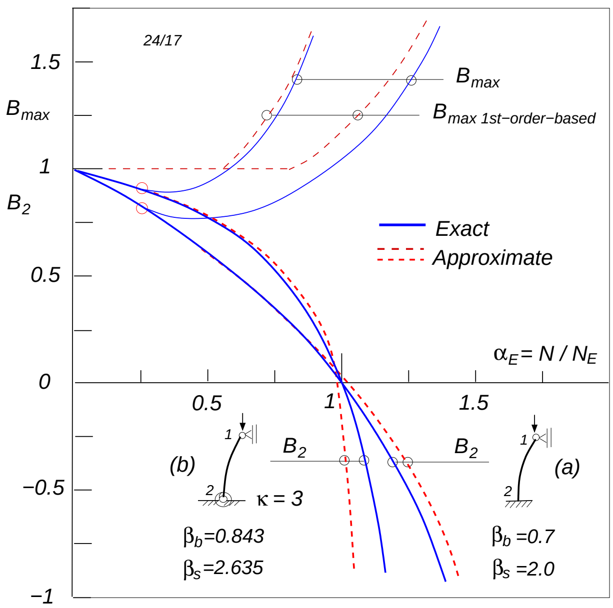

Figure 11: End moment and maximum moment magnifiers versus axial load level ($p = 8, q = 0.6$).

End moments. It is of interest to briefly study the ability of the presented approach to reflect local second order effects on end moments. Laterally restrained columns, pinned at end 1, and rotationally restrained at end 2 is considered. From moment equilibrium, the end moment can be written $-M_2 = N\Delta + VL$, where $\Delta = B_s\Delta_0$ and the shear V is given by Eq. (4). Substituting for N_{cs}, Eq. (5), the end moment may be written

$$\frac{M_2}{B_s M_{02}} = B_2 = 1 - \frac{\gamma_n - 1}{\gamma_s} \cdot \frac{N}{N_{cs}} \tag{44}$$

The same equation results for a symmetrically restrained column, with equal end moments.

Approximate results by Eq. (44) are compared to exact elastic results in Fig. 11 for two columns pinned at the top, and either (a) clamped or (b) elastically restrained at the base. The parameter $\alpha_{s,b} = (\beta_s/\beta_b)^2$ (see Eq. (9 b) and (10)) are 8.13 and 9.77, respectively, for these columns. The agreement in the negative moment range is not so good for the latter column, but on the overall, and particularly in the most important, positive moment range, the agreement is good.

For a general column, however, with unequal end restraints, the end moments above must be replaced by the sums of the two end moments. The distribution of the sum to the two ends must be established before individual end moments can be calculated. This is not straightforward, and remains a task for future research. A limited, simplified approach has been discussed elsewhere [8].

Maximum moments. Exact maximum moment magnifiers B_{max} are also shown in the figure. They are seen to form away from column end 2 for loads in excess of α_E=0.25 in these particular cases. Such magnifiers can readily be determined [22, 37, 38] based on exact end moments. Until an approximate theory for end moment predictions in the general case with unequal end restraints become available, approximate maximum moment computations will have to be based on approximations such as

$$B_{max} = \frac{C_m}{1 - N/N_{cb}} \geq 1.0 \quad ; \quad C_m = 0.6 + 0.4\mu_o \quad (\geq 0.4) \qquad (45\text{a, b})$$

where

$$\mu_o = \frac{M_{01}}{M_{02}} = \frac{M_{01b} + B_s\, M_{01s}}{M_{02b} + B_s\, M_{02s}} \qquad (46)$$

is a ratio defined in Hellesland [21] between the smaller and larger "sway-modified first-order end moments". The ratio is taken positive for members bent in single curvature, and negative otherwise. Subscript "b" indicates "braced moments", obtained when the frame is considered fully braced, and "s" indicates "sway moments" of the real frame due to the sidesway.

Eq. (45) is a most common magnifier and adopted by many codes (e.g., [24, 28, 33]), with or without the limit 0.4 on C_m, but normally with the end moment ratio μ_o defined with first-order end moments of the column considered braced. From a mechanics point of view, the definition above (Eq. (46)) is more correct [21].

Predictions by Eq. (45), with μ_o=0 for the two columns in Fig. 11, are shown (dashed lines) in the figure. The predictions are very conservative at low to intermediate load levels.

12 Summary and conclusions

Analysis tools have been presented for unbraced or partly braced frames with sway. They will enable reasonably accurate predictions, or more simplified predictions, depending on required accuracy. They may be useful in both preliminary analysis and design, or for carrying out independent checks of results obtained by more advanced computer methods.

General sway magnifier and critical load formulations that account for global and local second order effects have been derived, and simplifications leading to various existing, simplified formulations and code adaptions have been presented and discussed.

The formulations are presented both in terms of lateral first order storey stiffness and the sum of critical, free sway column loads, and it is shown that these formulations are derived on the same basis, and give the exact same predictions for the same end restraints. This is emphasised, since it does not always seem to be well understood, and since the impression is sometimes given in some relevant literature that the two formulations may be based on different premises.

The general formulations include a proposed, approximate, higher order shear-axial load relationship that includes local second order effects and in which the main element is an extended flexibility factor expression. Use of this factor, that may vary between unity and very large values, provides an improved understanding of the mechanics of the transition from unbraced to braced bending modes with increasing load level.

Predictions using the extended approach compare well with exact critical loads, sway and moment magnifiers. Various simplifications have been considered. Predictions using a simplified flexibility factor that varies in the range 1-1.216, provide quite acceptable accuracies for load levels below about 60% of the critical load.

Acknowledgement

The author became interested in the subject area in conjunction with a specific bridge project (1973-75), across a strait crossing requiring columns of different lengths, while working in the structural engineering firm of Dr. Ing. A. Aas-Jakobsen (AAJ), Oslo, Norway. The present work represents an extension of that earlier work [8].

References

[1] Rosenblueth, E. Slenderness effects in buildings. J. Struct. Division, ASCE 1965;91(1):229–252.

[2] Fey, T., Vereinfachte Berechnung von Rahmensystemen des Stahlbetonbaus nach der Theorie 2. Ordnung (Approximate second order analysis of reinforced concrete frames). Der Bauingenieur, 1966;41(6):231–238.

[3] Parme, A. Capacity of restrained eccentrically loaded long columns. Symposium on Reinforced Concrete Columns, Special Publication, American Concrete Institute (ACI), Detroit, 1966;SP-13:325–367.

[4] Stevens, L. K. Elastic stability of practical multi-storey frames. Proceedings, Institution of Civil Engineers, 1967;36:99–117.

[5] Rubin, H. Das Q_Δ-Verfahren zur vereinfachten Berechnung verschieblicher Rahmensysteme nach dem Traglastverfahren der Theorie II. Ordnung. Der Bauingenieur, 1973;48(8):275–285.

[6] Horne, M. R. An approximate method for calculating the elastic critical loads of multi-storey plane frames. The Struct. Eng., 1975;53(6):242–248.

[7] Wood, B. R., Beaulieu, D., Adams, P. F.. Column design by P Delta method. J. Struct. Division, ASCE 1976;102(2):411–427.

[8] Hellesland J. Approximate second order analysis of unbraced frames. Technical Rep. Oslo (Norway): Dr. Ing. A. Aas-Jakobsen Ltd.; 1976. 43 pp.

[9] LeMessurier, W.M.. A practical method of second order analysis. Eng. J., AISC 1977;14(2):49–67.

[10] MacGregor J.G., Hage, S. E. Stability analysis and design of concrete frames. J. Struct. Division, ASCE 1977;103(10):1953–1970.

[11] Lai, S.-M.A., MacGregor, J.G. Geometric nonlinearities in unbraced frames. J. Struct. Eng., ASCE 1983;109(11):2528–2545.

[12] Aristizabal-Ochoa JD. K-factor for columns in any type of construction: Nonparadoxical approach. J. Struct. Eng., ASCE 1994;120(4):1272-1290.

[13] Aristizabal-Ochoa, J. D. Story stability and minimum bracing in RC framed structures: A general approach. ACI Struct. J. 1995;92(6):735–744.

[14] Lui, E. M. A novel approach for K factor determination. Eng. J., AISC 1992;29(4):150–159.

[15] Xu, L., Liu, Y. Story stability of semi-braced steel frames. J. Construct. Steel Res., 2002;58(4):467–491.

[16] Xu, L., Liu, Y. Story-based effective length factors of unbraced PR frames. Eng. J., AISC 2002;39(1):13–29.

[17] Girgin K, Ozmen G, Orakdogen E. Buckling lengths of irregular frame columns. J. Construct. Eng., 2006; 62(6):605–613.

[18] Chen, WF, Lui, EM. Stability design of steel frames. Boca Raton/Ann Arbor/Boston/London: CRC Press; 1991.

[19] Hellesland J. Approximate second order analysis of unbraced frames reflecting interstorey interaction in single curvature regions. Research Rep. in Mechanics, No. 08-2. Oslo (Norway): Mechanics Division, Dept. of Math., Univ. of Oslo; 2008.

[20] Hellesland J, Bjorhovde R. Restraint demand factors and effective lengths of braced columns. J. Struct. Eng., ASCE 1996;122(10):1216-1224.

[21] Hellesland, J. Mechanics and effective lengths of columns with positive and negative end restraints. Eng. Struct., 2007; 29(12):3464–3474.

[22] Lai, S.-M. A., MacGregor, J.G., Hellesland, J. Geometric nonlinearities in nonsway frames. J. Struct. Eng., ASCE 1983;109(12):2770–2785.

[23] Wang CK. Intermediate structural analysis. New York (NY):McGraw-Hill, Inc.; 1983.

[24] ANSI/AISC 360-05. Specifications for structural steel buildings, and Commentary. American Institute of Steel Construction (AISC), Chicago, Ill.; 2005.

[25] Hellesland J, Bjorhovde R. Improved frame stability analysis with effective lengths. J. Struct. Eng., ASCE 1996;122(11):1275-1283.

[26] Hellesland J. Simplified system instability analysis. 4th Pacific Structural Steel Conference (PSSC'95), Singapore, 1995: Vol. 1, 95-102. Editors: NE Shanmugan and YS Choo, Pergamon Press.

[27] Hellesland J. Application of the method of means to the stability analysis of unbraced frames. J. Construct. Steel Res., 1998;46(1-3):147. Full paper available on CD-ROM, paper 147. Also available in Mech. and Appl. Math. Series, No. 97-4, Dept.of Math., Univ. of Oslo, Norway; 1997. 14 pp.

[28] ACI Committee 318. Building code requirements for structural concrete (ACI 318-08) and Commentar. American Concrete Institute (ACI), Farmington Hills, Mich.; 2008.

[29] MacGregor J.G., Breen, J. E., Pfrang, E. O. Design of slender concrete columns. ACI J., 1970;67(1):6–28.

[30] Yura, JA. Effective lengths of columns in unbraced frames. Eng. J., AISC 1971;8(2):37–42.

[31] AS4100–1990. Steel structures. Sydney: Standards Australia; 1990.

[32] Eurocode 3: Design of steel structures – Part 1.1: General rules and rules for buildings (European standard EN 1993-1-1:2003:E). European Committee for Standardisation (CEN). Brussels; 2003.

[33] Eurocode 2: Design of concrete structures – Part 1-1: General rules and rules for buildings (EN 1992-1-1:2004:E). European Committee for Standardisation (CEN). Brussels; December 2004.

[34] Hajjar, JF, White, DW. The accuracy of column stability calculations in unbraced frames and the influence of columns with effective length factors less than one. Eng. J., AISC 1994;31(3):81–97.

[35] Hellesland, J. Discussion of 'An improved K-factor formula' by Cheong-Siat-Moy, F. J. Struct. Eng., ASCE 2000;126(5):633-635.

[36] Cheong-Siat-Moy, F. An improved K-factor formula. J. Struct. Eng., ASCE 1999;125(2):169–174.

[37] Hellesland J. Mechanics and slenderness limits of sway-restricted RC columns. J. Struct. Eng., ASCE 2008;134(8):xx-xx. (Also available in E-print series: Mechanics and Applied Mechanics, No. 07-1. Oslo (Norway): Dept. of Math., Univ. of Oslo; 2007. 21 pp.)

[38] Galambos TV. Structural members and frames. New York (NY):Prentice-Hall, Inc.; 1968.

Appendix–Example: Diagonal bracing

In the case of frames with external bracings, it is necessary to establish the stiffness S_B of the bracing. An example is shown here for the diagonal bracing shown in Fig. 12.

The axial tensile force in the tie is a function of the displacement Δ. Assuming small displacements, it can be expressed by

$$N_B = \sigma A = \varepsilon EA = \frac{\Delta \cos\theta}{L_B} EA \tag{47}$$

The bracing stiffness, defined by the horizontal component of N_B per unit lateral displacement, can then be expressed in any of the forms below.

$$S_B = \frac{N_B \cos\theta}{\Delta} = \frac{EA\cos^2\theta}{L_B} = \frac{EA \cdot L_b^2}{L_B^3} \tag{48}$$

The vertical component of N_B ($N_B \sin\theta$) will provide an axial load in the right column, and should be included as such if it is not negligible.

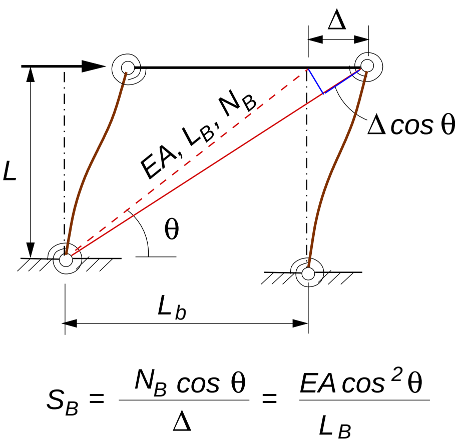

Figure 12: Diagonal bracing.